UK Junior Mathematical Olympiad 2015 Solutions

A1 20:15 The 225 minutes are equal to 3 hours and 45 minutes. At 3 hours before midnight the time is 21:00. So 45 minutes earlier than that the time is 20:15.

A2 23 A cubical die has six faces and so three pairs of opposite faces. The numbers on each pair of opposite faces add to 10, so the sum of the numbers on all six faces is 30. Therefore the sum of the numbers on the unseen faces is $30 - 7 = 23$.

A3 36 cm^2 The length of the perimeter of the shaded region is the same as the length of the perimeter of the square. Hence the square has side length 6 cm and so has area 36 cm^2.

A4 24 Suppose there are initially m apples and n oranges in the basket. Since $m : n = 3 : 8$, it follows that $8m = 3n$. After removing one apple, the ratio becomes $(m - 1) : n = 1 : 3$, and so $3(m - 1) = n$. Putting these two equations together,

$$8m = 3 \times 3(m - 1)$$

$$8m = 9m - 9$$

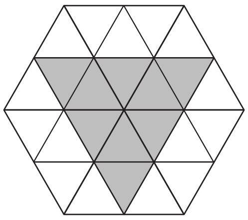
$$m = 9.$$

Using the second equation gives $n = 3 \times 8 = 24$. So there are 24 oranges in the basket.

A5 $(6 - \pi)$ cm^2 The area bounded by the circles can be calculated by subtracting the area of one circle from the area of the 3 cm × 2 cm rectangle with edges passing through the centres of the circle. Therefore the shaded area is $(6 - \pi)$ cm^2.

A6 40 Two players are knocked out as a result of one match. To leave a winner, 80 players must be knocked out. Therefore there must be 40 matches.

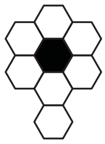
A7 63 cm The hexagon may be divided into small equilateral triangles as shown.



Each of these small triangles has side length 7 cm and so the perimeter of the shaded triangle is 9×7 cm $= 63$ cm.

A8 0 The sum can be factorised to give $9^{2015} + 9^{2016} = 9^{2015}(1 + 9)$. This is equal to $9^{2015} \times 10$ and so, since this number is a multiple of 10, its units digit must be 0.

A9 $\frac{1}{8}$ The tiling pattern can be divided into identical pieces, each of which has the following shape.

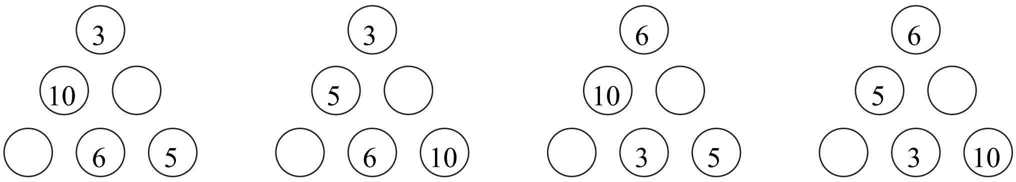


As $\frac{1}{8}$ of each piece is black, it follows that $\frac{1}{8}$ of the plane is black.

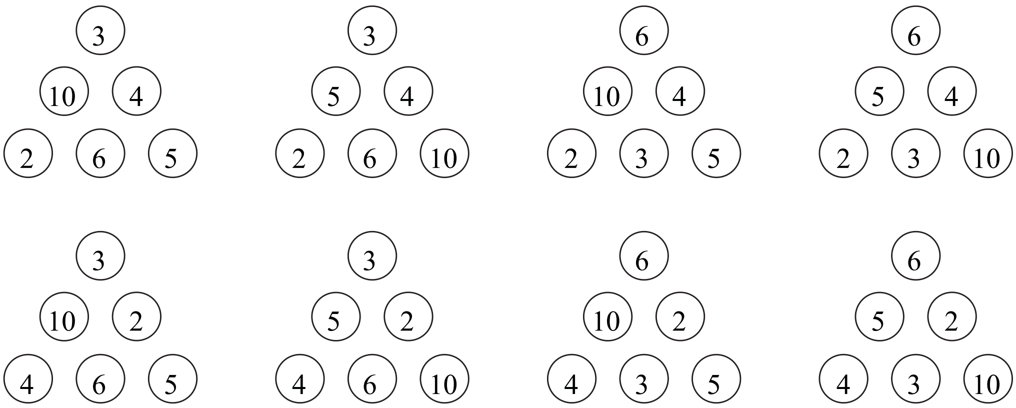
A10 **120** Each of the six numbers can be written as a product of primes as follows:

$$2, \quad 3, \quad 2 \times 2, \quad 5, \quad 2 \times 3, \quad 2 \times 5.$$

Since only two 3s appear in the list, the numbers 3 and 6 cannot appear in the same edge of the triangle; otherwise the products of the three numbers along this edge would have a factor of 3^2 and product of the three numbers along the other two edges would not. The same is true for 5 and 10 since there are only two 5s in the list. These observations mean that, up to symmetry, the only possible arrangements are:



For each of these arrangements, the 2 and the 4 can be placed in two ways to give the following eight arrangements.



Consider the products of the numbers along the edges of each of these arrangements. In the first arrangement, the products are all 60; in the last, the products are all 120; in each of the others, the products are not all equal. Therefore the largest possible product satisfying the conditions of the question is 120.

B1 Let N be the smallest positive integer whose digits add up to 2015.
What is the sum of the digits of $N + 1$?

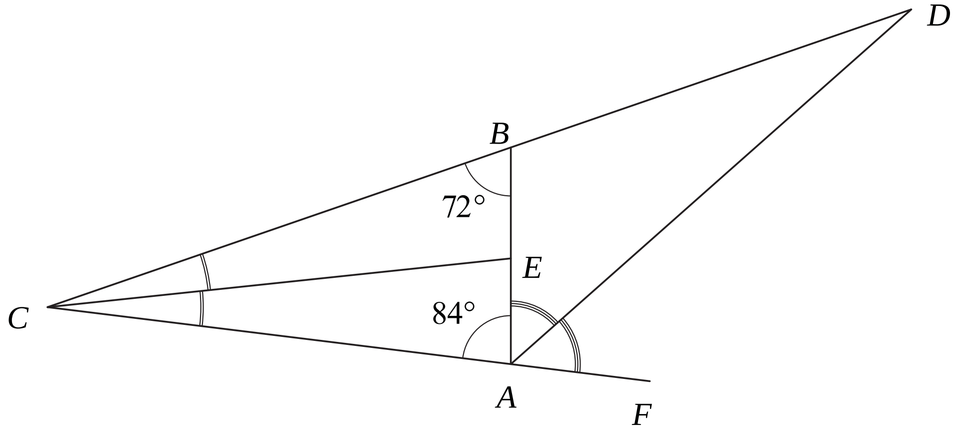
Solution

For such an integer to be as small as possible, it must have as few digits as possible (since any integer with more digits would be larger).

Since $\frac{2015}{9} = 223$ remainder 8, the smallest possible number of digits is 224. Thus any number whose digits are made up of 223 copies of '9' and one '8' will have the correct digit sum and use the smallest possible number of digits. The integer N must have '8' as its leading digit, followed by 223 copies of '9'; any other arrangement of these digits would give a larger integer.

Thus $N + 1 = 9 \times 10^{223}$. Hence the digit sum of $N + 1$ is 9.

B2 The diagram shows triangle ABC, in which $\angle ABC = 72°$ and $\angle CAB = 84°$. The point E lies on AB so that EC bisects $\angle BCA$. The point F lies on CA extended. The point D lies on CB extended so that DA bisects $\angle BAF$.



Prove that $AD = CE$.

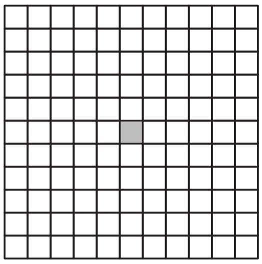
Solution

Look first at triangle ABC. Since the angles in a triangle add to 180°, $\angle BCA = 24°$. Hence, since CE bisects $\angle BCA$, $\angle BCE = \angle ECA = 12°$. Next consider triangle ECA and use the angle sum again to obtain $\angle AEC = 84°$. Therefore $\angle CAE = \angle AEC$ and so triangle ECA is isosceles and hence $AC = CE$.

Consider the angles at A. Since angles on a straight line add to 180°, $\angle BAF = 96°$. Because AD bisects $\angle BAF$, $\angle BAD = 48°$. Finally, consider triangle DCA. Since the angles in a triangle add to 180°, it must be the case that $\angle ADC = 24°$. Therefore $\angle BCA = \angle ADC$ and so triangle DCA is isosceles and hence $AC = AD$. Therefore $AD = CE$ since both are equal to AC.

B3 Jack starts in the small square shown shaded on the grid, and makes a sequence of moves. Each move is to a neighbouring small square, where two small squares are neighbouring if they have an edge in common. He may visit a square more than once.

Jack makes four moves. In how many different small squares could Jack finish?



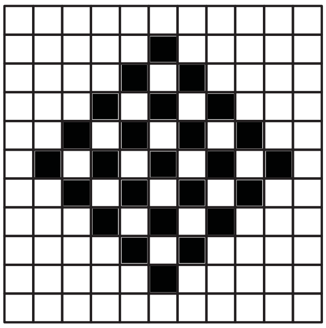
Solution

Let the starting square be labelled (0, 0) and the square to the right of the starting square be labelled (1, 0), and so on for the remaining squares in the grid.

Each move may be represented by $(+1, +0)$, $(+0, +1)$, $(-1, +0)$ or $(+0, -1)$ where, for example, $(+1, +0)$ represents a move of one square to the right and $(+0, -1)$ represents a move of one square downwards.

Each move increases or decreases the sum of the coordinates of the occupied square by 1. At (0, 0) the sum of the coordinates is 0. After one move, the sum of the coordinates must be 1 or -1. So after two moves the sum of the coordinates is 2, 0 or -2. After three moves, the sum of the coordinates of the occupied square is 3, 1, -1 or -3. Finally, after the fourth move, the sum of the coordinates of the occupied square is 4, 2, 0, -2, or -4.

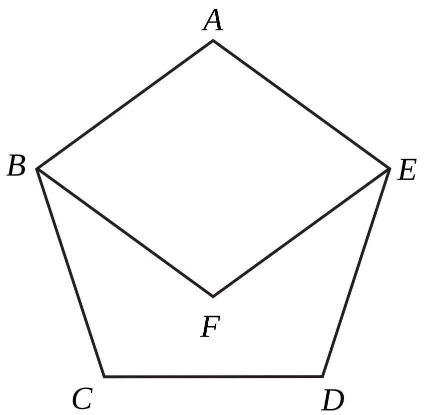
This means the sequence of four moves can end at any one of the 25 squares shown in black.



B4 The point F lies inside the regular pentagon $ABCDE$ so that $ABFE$ is a rhombus. Prove that EFC is a straight line.

Solution

Each interior angle of a regular pentagon is $108°$. The internal angles of a quadrilateral sum to $360°$ and so, since $ABFE$ is a rhombus, $\angle ABF = \angle FEA = 72°$. Therefore $\angle FBC = 36°$. Triangle FBC is isosceles since $BC = AB = BF$ and so $\angle BFC = \angle BCF = 72°$. Then $\angle EFC = \angle EFB + \angle BFC = 108° + 72° = 180°$ and so EFC is a straight line.



B5 I have two types of square tile. One type has a side length of 1 cm and the other has a side length of 2 cm.

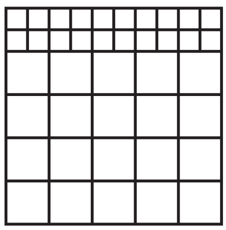
What is the smallest square that can be made with equal numbers of each type of tile?

Solution

Each of the 1 cm tiles has area 1 cm^2 and each of the 2 cm tiles has area 4 cm^2. Suppose there are n tiles of each type. Then the assembled square must have area $5n$ cm^2 and $5n$ must be a square number.

The smallest n could be is 5. This would require the tiles to fit together to form a 5 cm × 5 cm square. However the first four 2 cm tiles are placed in a 5 cm square, the fifth 2 cm tile cannot be placed in the remaining space. So n cannot be 5.

The next smallest n for which $5n$ is a square is 20. This would require the tiles to fit together to form a 10 cm × 10 cm square. This arrangement is possible; an example is shown below. Hence the smallest square that can be made is a 10 cm × 10 cm square.



B6 The letters a, b, c, d, e and f represent single digits and each letter represents a different digit. They satisfy the following equations:

$$a + b = d, \qquad b + c = e \qquad \text{and} \qquad d + e = f.$$

Find all possible solutions for the values of a, b, c, d, e and f.

Solution

The equations may be represented as the following triangle where each digit is the sum of the two adjacent digits directly above it.

$$\begin{array}{ccccc} a & & b & & c \\ & d & & e & \\ & & f & & \end{array}$$

None of the digits a, b, c, d and e may be 0 since this would force two of the others to be equal. The digit f cannot be 0 since $f = d + e$ and both d and e are positive.

Consider the middle row and note that both d and e must be at least 3 since they are each the sum of distinct positive integers. Now assume that $d < e$. Since $d \geqslant 3$ and $f \leqslant 9$ it follows that $e \leqslant 6$. This means that (d, e) is (3, 4), (3, 5), (3, 6) or (4, 5). However, the only way to write 4 as the sum of two different nonzero digits is 1 + 3; therefore $(d, e) \neq (3, 4)$.

Suppose $(d, e) = (3, 5)$.

$$\begin{array}{ccccc} a & & b & & c \\ & 3 & & 5 & \\ & & 8 & & \end{array}$$

Then $b \neq 2$ since this would mean $c = 3$. So it must be the case that $(a, b, c) = (2, 1, 4)$.

Now suppose $(d, e) = (3, 6)$.

$$\begin{array}{ccccc} a & & b & & c \\ & 3 & & 6 & \\ & & 9 & & \end{array}$$

Then a must be 1 or 2 and so (a, b, c) is (1, 2, 4) or (2, 1, 5).

Finally, suppose $(d, e) = (4, 5)$.

$$\begin{array}{ccccc} a & & b & & c \\ & 4 & & 5 & \\ & & 9 & & \end{array}$$

Then a must be 1 or 3. But a cannot be equal to 3 as this would force c to be 4. So $(a, b, c) = (1, 3, 2)$.

Therefore, with the assumption that $d < e$, there are four solutions. Another four can be obtained by assuming $d > e$, which has the effect of reflecting each of the triangles vertically.

The solutions for (a, b, c, d, e, f) are

(2, 1, 4, 3, 5, 8),	(1, 2, 4, 3, 6, 9),	(2, 1, 5, 3, 6, 9),	(1, 3, 2, 4, 5, 9),
(4, 1, 2, 5, 3, 8),	(4, 2, 1, 6, 3, 9),	(5, 1, 2, 6, 3, 9),	(2, 3, 1, 5, 4, 9).