

## UK Junior Mathematical Olympiad 2017 Solutions

**A1. 100203.4** When you convert all the distances to cm, you obtain 1 km = 100000 cm, 2 m = 200 cm and 4 mm = 0.4 cm. Therefore 1 km 2 m 3 cm 4 mm is equal to 100203.4 cm.

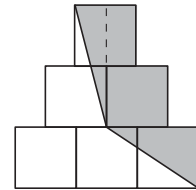
**A2. 20** The volume of the large cube is  $27 \text{ cm}^3$ . To make the hole, one small cube is removed from the centre of each face and one from the centre of the large cube. Each small cube has a volume of  $1 \text{ cm}^3$ . Hence seven small cubes are removed and the remaining volume is  $20 \text{ cm}^3$ .

**A3.  $\frac{4}{5}$**  Howard has completed  $\frac{1}{2}$  of the run. He is now  $\frac{3}{5}$  of the way through the second half. Hence he has completed  $\frac{1}{2} + \frac{3}{5} \times \frac{1}{2} = \frac{1}{2} + \frac{3}{10} = \frac{4}{5}$  of the whole run.

**A4. £1729** Since  $2017 = 288 \times 7 + 1$ , you can buy 288 lots of 7 bookmarks at £6 each. Hence the smallest amount you could pay for 2017 of the bookmarks is  $£6 \times 288 + £1 \times 1 = £1729$ .

**A5. 9** Since 13 days and 22 hours are equivalent to 334 hours, the yacht travels 3000 nautical miles in 334 hours. Therefore the yacht travels  $\frac{3000}{334}$  nautical miles in 1 hour. Hence the yacht's average speed in nautical miles per hour is 9 (to the nearest integer).

**A6.  $\frac{11}{24}$**  There are several ways to solve this problem. This solution is just one example. Without loss of generality, let each square have a side length of 2 units. Hence the six identical squares have a total area of  $6 \times 2^2 = 24$  square units. The grey shaded area can be thought of as consisting of two triangles, one rectangle and one square, as shown on the diagram. Therefore the grey shaded area is



$$\frac{1}{2} \times 1 \times 4 + \frac{1}{2} \times 3 \times 2 + 1 \times 2 + 2^2 = 11 \text{ square units.}$$

Hence  $\frac{11}{24}$  of the diagram is shaded.

**A7. 95%** The required percentage is  $\frac{9}{9.5} \times 100 = \frac{18}{19} \times 100 = \frac{1800}{19} \approx 94.7$  by long division. This is 95% to the nearest 1%.

**A8. 6 : 5** Let  $m$  be the original number of marbles in the jar. Therefore, as Jenny replaces 2 of the white marbles with 7 red marbles, there are now  $m + 5$  marbles in the jar. We know that  $\frac{4}{5}$  of the original number of marbles were white, that 2 white marbles were removed and that now  $\frac{3}{5}$  of the jar's marbles are white. Hence

$$\frac{4}{5}m - 2 = \frac{3}{5}(m + 5)$$

Solving this equation, gives  $m = 25$ . Therefore the ratio of the total number of marbles in the jar now to the number in the jar before is  $30 : 25 = 6 : 5$ .

**A9. 200** There are 333 multiples of 3 less than 1000, and there are 111 multiples of 9 less than 1000. As numbers that are multiples of both 3 and 10 are multiples of 30, consider the 33 multiples of 30 that are less than 1000. The lowest common multiple of 9 and 30 is 90 and there are 11 multiples of 90 less than 1000. Hence the number of multiples of 3 that are less than 1000 but not divisible by either 9 or by 10 is  $333 - 111 - 33 + 11 = 200$ .

**A10. 4, 1** Let the radius of the smaller circle be  $r$ , so the radius of the larger circle is  $3r$ . The area of the smaller circle is  $\pi r^2$  and the grey shaded area is  $\frac{1}{2}\pi(3r)^2 - \frac{1}{2}\pi r^2 = 4\pi r^2$ . Therefore the ratio of the grey shaded area to the area of the smaller circle is  $4\pi r^2 : \pi r^2 = 4 : 1$ , and hence  $a = 4$  and  $b = 1$ .

- B1.** An amount of money is to be divided equally between a group of children. If there was 20p more than this amount, then there would be enough for each child to receive 70p. However, if each child was to receive 60p, then £2.10 would be left over. How many children are there in the group?

*Solution*

Let  $C$  be the number of children in the group and let  $A$  be the total amount of money in pence to be divided between the children.

Then  $\frac{A + 20}{C} = 70$ , so that  $A + 20 = 70C$  and therefore  $A = 70C - 20$ .

Also  $A = 60C + 210$ .

Hence  $70C - 20 = 60C + 210$ .

Solving this last equation we obtain  $C = 23$ . Therefore there are 23 children in the group.

It is good practice to check the solution works. The total amount of money is £15.90, and  $(£15.90 + £0.20) \div 23 = £0.70$ ;  $£0.60 \times 23 = £13.80$ , and  $£13.80 + £2.10 = £15.90$ .

- B2.** A 3-digit integer is called a ‘V-number’ if the digits go ‘high-low-high’ – that is, if the tens digit is smaller than both the hundreds digit and the units (or ‘ones’) digit. How many 3-digit ‘V-numbers’ are there?

*Solution*

The smallest ‘V-number’ is 101 and the largest ‘V-number’ is 989.

Consider the tens digits. The smallest tens digit is 0 and the largest tens digit is 8.

If the tens digit is 0, the hundreds digit can be 1 to 9, and the units digit can be 1 to 9, giving  $9 \times 9$  possible ‘V-numbers’.

If the tens digit is 1, then the hundreds digit can be 2 to 9 and the units digit can be 2 to 9, giving  $8 \times 8$  possible ‘V-numbers’.

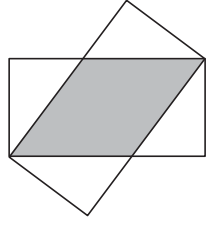
If the tens digit is  $d$ , where  $d$  can be any digit from 0 to 8, the hundreds digit can be  $(d + 1)$  to 9 and the units digit can be  $(d + 1)$  to 9, giving  $(9 - d) \times (9 - d)$  possible ‘V-numbers’.

The greatest value of  $d$  is 8. In this case, the hundreds digit can only be 9 and the units digit can only be 9, which gives just  $1 \times 1$  possibilities.

This gives the total number of possible ‘V-numbers’ to be  $9 \times 9 + 8 \times 8 + \dots + 1 \times 1 = 285$ , which is the sum of the squares from 1 to 9 inclusive.

- B3.** Two identical rectangles overlap in such a way that a rhombus is formed, as indicated in the diagram. The area of the rhombus is five-eighths of the area of each rectangle.

What is the ratio of the length of the longer side of the rectangle to the length of the shorter side?



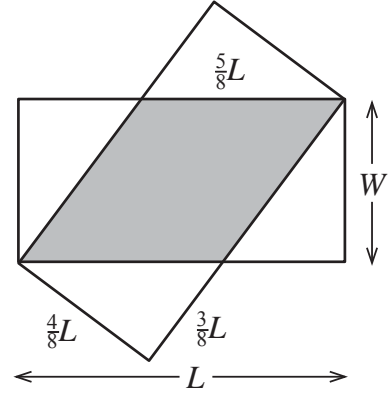
*Solution*

Let the length of the longer side and of the shorter side of the rectangle be  $L$  and  $W$  respectively.

Since the area of the rhombus is  $\frac{5}{8}$  of the area of each rectangle, the area of the rhombus is  $\frac{5}{8}LW$ . Also, since the area of a rhombus is equal to base  $\times$  perpendicular height and the perpendicular height of the shaded rhombus is  $W$ , the length of each side of the rhombus is  $\frac{5}{8}L$ .

Consider one of the white right-angled triangles.

The length of the hypotenuse is  $\frac{5}{8}L$  and the length of one other side is  $L - \frac{5}{8}L = \frac{3}{8}L$ . Therefore, using Pythagoras' Theorem, we can find the length of the third side in terms of  $L$  since the triangle is a 3: 4: 5 triangle. Hence  $W = \frac{4}{8}L$  and the ratio of the length of the longer side of the rectangle to the length of the shorter side of the rectangle is  $L : \frac{4}{8}L = 2 : 1$ .



- B4.** My uncle lives a long way away and his letters always contain puzzles. His three local teams are the Ants (A), the Bees (B), and the Cats (C), who play each other once a year. My uncle claimed that the league table part way through the year looked like this:

	Played	Won	Drawn	Lost	Goals for	Goals against
A	1	0	0	1	4	2
B	2	1	1	0	2	2
C	2	1	0	1	3	1

When we complained that this is impossible, he admitted that every single number was wrong but he excused himself because every number was exactly '1 out'.

Find the correct table, explaining clearly how you deduced the corrections.

#### *Solution*

The maximum number of games any team can play is 2, as each team only plays another team once in the year and there are only 3 teams.

Therefore team B (the Bees) and team C (the Cats) played 1 game each.

Team A (the Ants) played 2 games (because if they had played 0 games they would have 0 goals for). Since each figure is '1 out', team A won 1, drew 1 and lost 0 (so that the total number of matches played is 2).

Since team A won 1 match, team B lost 1 match (since team B had lost 0 games originally and team C cannot have lost 2 games). Therefore, team B won 0 games and drew 0 games.

So team A drew against team C. Therefore the number of games resulting in a draw for team C is 1 and, as they only played 1 match, they won and lost 0 games.

Since team C's only game resulted in a draw, team C's goals for and against are equal. Therefore, team C's goals for and against are 2 each.

Team B's only match resulted in a loss, so team B's goals against are greater than its goals for. Therefore, the number of team B's goals for is 1 and the number of its goals against is 3.

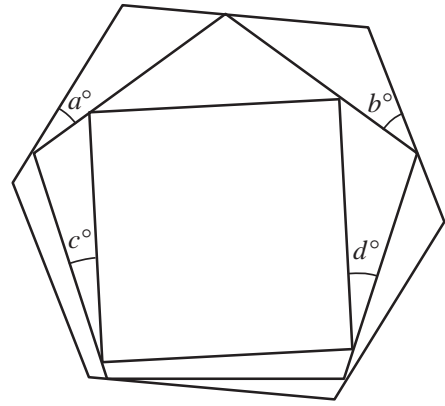
Because team B and team C both only played team A, the number of team A's goals for is equal to the sum of the number of team B's goals against and the number of team C's goals against. Hence the number of team A's goals for is 5. Similarly, the number of team A's goals against is the sum of the number of team B's goals for and the number of team C's goals for. Therefore the number of team A's goals against is 3.

So the correct table is

Team	Played	Won	Drawn	Lost	Goals for	Goals against
A	2	1	1	0	5	3
B	1	0	0	1	1	3
C	1	0	1	0	2	2

- B5.** The diagram shows a square whose vertices touch the sides of a regular pentagon. Each vertex of the pentagon touches a side of a regular hexagon.

Find the value of  $a + b + c + d$ .



*Solution*

Each interior angle of a regular pentagon is  $108^\circ$  and each interior angle of a regular hexagon is  $120^\circ$ .

Consider the two shaded triangles in the diagram alongside, which contain the angles  $a^\circ$  and  $b^\circ$  and the side of the hexagon that these triangles have in common.

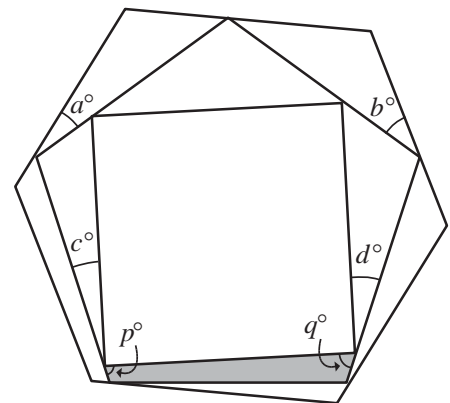
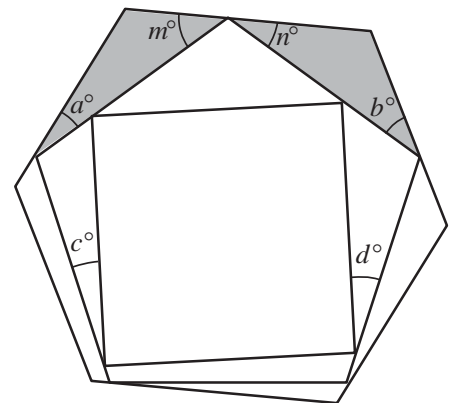
Let the two angles shown be  $m^\circ$  and  $n^\circ$ . Since the sum of the interior angles in a triangle is  $180^\circ$ , we have  $120 + a + m = 180$  and  $120 + b + n = 180$ . Therefore,  $m = 60 - a$  and  $n = 60 - b$ . Now, since the sum of the angles on a straight line is  $180^\circ$ , we have  $m + 108 + n = 180$ . Hence  $60 - a + 108 + 60 - b = 180$ . Therefore  $a + b = 48$ .

The region outside the square but inside the pentagon consists of a quadrilateral, shown shaded in the diagram alongside, and three triangles. We now consider the quadrilateral.

Let the two angles shown be  $p^\circ$  and  $q^\circ$ . Since the interior angles of a square are  $90^\circ$  and the sum of the angles on a straight line is  $180^\circ$ , we have  $c + 90 + p = 180$  and  $d + 90 + q = 180$ . Therefore  $p = 90 - c$  and  $q = 90 - d$ . Now, since the sum of the angles in a quadrilateral is  $360^\circ$  and two of these angles are interior angles of a regular pentagon, we have

$108 + p + q + 108 = 360$ . Hence  $216 + 90 - c + 90 - d = 360$ . Therefore  $c + d = 36$ . Hence

$$\begin{aligned} a + b + c + d &= 48 + 36 \\ &= 84. \end{aligned}$$



**B6.** The 9-digit positive integer  $N$  with digit pattern  $ABCABCBBB$  is divisible by every integer from 1 to 17 inclusive.

The digits  $A$ ,  $B$  and  $C$  are distinct. What are the values of  $A$ ,  $B$  and  $C$ ?

*Solution*

Since  $N$  is divisible by both 2 and 5,  $N$  is divisible by 10 and hence  $B = 0$ . Therefore  $N$  is of the form  $A0CA0C000 = A0C \times 1001 \times 1000$ .

Now  $1001 = 7 \times 11 \times 13$  and  $1000 = 2 \times 2 \times 2 \times 5 \times 5 \times 5$ . Hence  $A0CA0C000$  is certainly divisible by 1, 2, 4, 5, 7, 8, 11, 13 and 14.

We are told that  $N$  is divisible by every integer from 1 to 17. Hence, in particular,  $N$  is divisible by 9. Therefore, since the rule for divisibility by 9 is that the sum of the digits of the number is also divisible by 9, we have  $2A + 2C$  is a multiple of 9 and hence  $A + C$  is a multiple of 9. Also, since  $A$  and  $C$  are distinct,  $A + C = 9$ .

Once  $A$  and  $C$  are chosen so that  $N$  is divisible by 9,  $N$  will also be divisible by 3, 6, 12 and 15.

Since  $N = A0C \times 1001 \times 1000$  is divisible by 16,  $A0C$  is divisible by 2 and hence  $C$  is even. Therefore, the only options for  $A$  and  $C$  (in that order) are 1 and 8, 3 and 6, 5 and 4, or 7 and 2.

To ensure  $N$  is divisible by 17, we must now ensure that  $A$  and  $C$  are chosen so that  $A0C$  is divisible by 17. When we look at each case in turn, we find that  $108 = 6 \times 17 + 6$ ,  $306 = 18 \times 17$ ,  $504 = 29 \times 17 + 11$  and  $702 = 41 \times 17 + 5$ . Therefore  $A = 3$ ,  $B = 0$  and  $C = 6$ .